

# On $K_2$ of 1-dimensional local rings

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## Abstract

We study  $K_2$  of one-dimensional local domains over a field of characteristic 0, introduce a conjecture, and show that this conjecture implies Geller's conjecture. We also show that Berger's conjecture implies Geller's conjecture, and hence verify it in many new cases.

## 1 Introduction

Let  $A$  be an one-dimensional local integral domain which is essentially of finite type over a field  $K$  of characteristic zero. Let  $F$  denote the field of fractions of  $A$ . It is easy to check that the map  $K_1(A) \rightarrow K_1(F)$  is injective. It was a general question if the map  $K_2(A) \rightarrow K_2(F)$  is also injective, which is now known not to be true in general. Quillen's proof of Gersten's conjecture implies that this is true if  $A$  is a regular local ring. Dennis and Sherman ([G]) showed that this map is not injective when  $A$  is the local ring of the singular point of the cuspidal curve  $\text{Spec}(K[t^2, t^3])$ . The general picture about singular rings is given by the following conjecture of Geller ([G]).

**Conjecture. (Geller)** *Let  $A$  be a local one-dimensional domain with field of fractions  $F$ . Then  $A$  is regular if and only if the map  $K_2(A) \rightarrow K_2(F)$  is injective.*

This conjecture was verified by Geller ([G]) when  $A$  is noetherian, equicharacteristic, characteristic zero, and is also seminormal with finite normalisation. In the same article, Dennis and Sherman verify it for cuspidal rings of the type  $K[t^2, t^3]$  as described above. The conjecture is still unknown in almost all other cases. Our first aim in this paper is to formulate an Artinian version of this conjecture, and to show that this conjecture implies Geller's conjecture. Before we state the conjecture, let us recall that an algebra over a field  $K$  is called a *principal ideal algebra* if every ideal of  $A$  is principal. We call  $A$  to be finite-dimensional if it is finite over  $K$ . In this paper, we will have standing assumption that  $K$  is an algebraically closed field of characteristic zero, and all

$K$ -algebras are essentially of finite type over  $K$ . Our Artinian version of above conjecture is

**Conjecture. (AGC)** *If  $A$  is a subalgebra of a finite-dimensional principal ideal  $K$ -algebra  $B$  such that the map  $K_2(A) \longrightarrow K_2(B)$  is injective, then  $A$  is also a principal ideal algebra.*

We shall call this ‘Artinian Geller Conjecture’ (AGC). Our first result in this paper states

**Theorem 1.1** *With  $A$  and  $K$  as above, Artinian Geller Conjecture implies Geller’s Conjecture.*

In the other part of this paper, our aim is to relate these conjectures with differential forms in order to verify Geller’s conjecture in some cases. In this regard, we recall a similar conjecture about the module of Kahler differentials on one-dimensional local domain over a field of char. 0.

**Conjecture. (Berger)** *Let  $A$  be an one-dimensional local domain which is essentially of finite type over a field  $K$  of characteristic zero. Then  $A$  is regular if and only if the module of Kahler differentials  $\Omega_{A/K}$  is torsion-free.*

This conjecture was formulated by R. Berger in [B1] almost forty years ago. This has been verified in many cases (listed below) by various people though it is still unknown in general.

**Theorem 1.2** *If  $A$  is as above, with  $K$  algebraically closed, then Berger’s Conjecture implies Geller’s Conjecture.*

Before we state our corollary to this theorem, we recall that ([B2]) a local ring  $A$  as above is called an ‘almost’ complete intersection if the first quadratic transform of  $A$  is a complete intersection. The common examples are local rings of plane curves or a curve through a smooth point of a surface.

**Corollary 1.3** *Geller’s Conjecture is true in each of the following cases.*

- (i)  $A$  is seminormal (also proved by Geller),
- (ii)  $\mathfrak{M}^3 B \subset A$ , where  $B$  is the normalisation of  $A$  with Jacobson radical  $\mathfrak{M}$ ,
- (iii)  $A$  is a complete intersection,
- (iv)  $A$  is almost complete intersection,
- (v)  $A$  is the local ring of the vertex of an 1-dimensional graded ring with vertex as only singular point,

- (vi)  $A$  has analytically smoothable curve singularities,
- (vii)  $A$  has multiplicity  $< \binom{m}{2}$ , where  $m$  is the embedding dimension of  $A$ , and
- (viii)  $A$  has deviation  $\leq 3$ .

**Remark.** We mention here that the condition of the field  $K$  being algebraically closed is only a technical one and one can reduce the general case to this case using the techniques of [G] and [CGW]. In fact, it is shown in [CGW] that one can always assume  $K$  to be algebraically closed to prove Berger's conjecture.

## 2 Some results on Hochschild and Cyclic homology

In this section, we aim to prove some results concerning Hochschild and cyclic homology of rings. We refer the reader to [LO] for basic notions of Hochschild and Cyclic homology of rings. Let  $k$  be field of characteristic 0, and we assume all  $k$ -algebras to be commutative. For any  $k$ -algebra maps  $A \longrightarrow B$ , Loday ([LO]) also defines the relative Hochschild homology  $HH_*^k(A, B)$  over  $k$  as the homology groups of the chain complex  $\text{Cone}(C_\bullet(A) \longrightarrow C_\bullet(B))$ , where  $C_\bullet(A)$  denotes the Hochschild complex of  $A$  etc. For an ideal  $I$  of  $A$ ,  $HH_*^k(A, I)$  will be the relative homology of  $A$  and  $A/I$ . One defines relative Cyclic homology in similar way by taking the cone over the total cyclic complexes of the two algebras. We also have the notion of relative  $K$ -theory as defined, for example in [CS]. There are Chern class maps  $K_i(A) \longrightarrow HH_i^k(A)$ , (Dennis trace maps) and by functoriality of fibrations of  $K$ -theory spectra and Hochschild homology, one also has Chern class maps from relative  $K$ -theory to relative Hochschild homology ([LO]), which are compatible with long exact sequence of relative  $K$ -theory and Hochschild homology. It is known that there are natural maps  $\Omega_{A/k}^i \longrightarrow HH_i^k(A)$  and  $HH_i^k(A) \longrightarrow \Omega_{A/k}^i$  such that the composite is multiplication by  $i!$ . In particular,  $HH_1^k(A)$  is same as the module of Kahler differentials on  $A$  over  $k$ . For an ideal  $I$  of  $A$ , let  $\Omega_{(A,I)/k}^1 := \text{Ker}(\Omega_{A/k}^1 \twoheadrightarrow \Omega_{A/I/k}^1)$ . We begin with the following

**Lemma 2.1** *Let  $A$  be a  $k$ -algebra which is reduced and and is essentially of finite type over  $k$ . Let  $B$  be the normalisation of  $A$ , and let  $I$  be a conducting ideal for this normalisation. Then, for all sufficiently large  $n$ , the map*

$$HH_1^k(A, I^n) \longrightarrow HH_1^k(B, I^n)$$

*is injective.*

**Proof.** We use the following commutative diagram of exact sequences.

$$\begin{array}{ccccccc}
0 & \rightarrow & \frac{HH_2^k(A/I^n)}{HH_2^k(A)} & \rightarrow & HH_1^k(A, I^n) & \rightarrow & \Omega_{(A,I)/k}^1 \rightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & \frac{HH_2^k(B/I^n)}{HH_2^k(B)} & \rightarrow & HH_1^k(B, I^n) & \rightarrow & \Omega_{(B,I)/k}^1 \rightarrow 0
\end{array}$$

Diagram 1

It is enough to show that the vertical maps on the ends are injective. Put  $\overline{\Omega_{A/k}^1} := \text{Ker}(\Omega_{A/k}^1 \longrightarrow \Omega_{B/k}^1)$ . This module is supported on  $V(A/I)$  and hence is annihilated by  $I^n$  for  $n \gg 0$ . Thus,

$$I^n(\overline{\Omega_{A/k}^1}) = 0 \text{ for } n \gg 0. \quad (2.1)$$

Furthermore, since  $\Omega_{(A,I^n)/k}^1 = I^n\Omega_{A/k}^1 + d(I^n)$ , one has a diagram of exact sequences

$$\begin{array}{ccccccc}
0 & \rightarrow & d(I^n) & \rightarrow & \Omega_{(A,I^n)/k}^1 & \rightarrow & \frac{I^n\Omega_{A/k}^1}{d(I^n) \cap I^n\Omega_{A/k}^1} \rightarrow 0 \\
& & \parallel & & \downarrow & & \downarrow \\
0 & \rightarrow & d(I^n) & \rightarrow & \Omega_{(B,I^n)/k}^1 & \rightarrow & \frac{I^n\Omega_{B/k}^1}{d(I^n) \cap I^n\Omega_{B/k}^1} \rightarrow 0.
\end{array}$$

This gives

$$\text{Ker}(\Omega_{(A,I^n)/k}^1 \longrightarrow \Omega_{(B,I^n)/k}^1) = \text{Ker}\left(\frac{I^n\Omega_{A/k}^1}{d(I^n) \cap I^n\Omega_{A/k}^1} \longrightarrow \frac{I^n\Omega_{B/k}^1}{d(I^n) \cap I^n\Omega_{B/k}^1}\right).$$

Next, we

**Claim.** For  $n \gg 0$ ,  $I^n\Omega_{A/k}^1 \hookrightarrow I^n\Omega_{B/k}^1$ .

To prove the claim, notice that  $\Omega_{A/k}^1$  is a finitely generated  $A$ -module, and hence by Artin-Rees theorem, there exists  $c > 0$  such that for all  $n > c$ ,

$$(I^n\Omega_{A/k}^1 \cap \overline{\Omega_{A/k}^1}) \subset I^{n-c}(I^c\Omega_{A/k}^1 \cap \overline{\Omega_{A/k}^1}).$$

In particular, we get

$$(I^n\Omega_{A/k}^1 \cap \overline{\Omega_{A/k}^1}) \subset I^{n-c}(\overline{\Omega_{A/k}^1}) = 0 \text{ for } n \gg 0 \text{ by 2.1,}$$

which proves the claim.

Using this claim, we obtain

$$\text{Ker}\left(\frac{I^n\Omega_{A/k}^1}{d(I^n) \cap I^n\Omega_{A/k}^1} \longrightarrow \frac{I^n\Omega_{B/k}^1}{d(I^n) \cap I^n\Omega_{B/k}^1}\right) = \frac{I^n\Omega_{A/k}^1 \cap (d(I^n) \cap I^n\Omega_{B/k}^1)}{d(I^n) \cap I^n\Omega_{A/k}^1}$$

$$= \frac{d(I^n) \cap I^n \Omega_{A/k}^1}{d(I^n) \cap I^n \Omega_{A/k}^1} = 0.$$

This, together with 2.1 implies that the right-most vertical map in Diagram 1 is injective for all sufficiently large  $n$ .

Before we start proving the injectivity of the vertical map on the left, we make the convention that all Hochschild and cyclic homologies will be considered over the given base field  $k$  in the remaining part of this lemma, and we will suppress this field  $k$ . We use the Hodge decomposition (or  $\lambda$ -decomposition) ([LO] or [C]) on Hochschild homology to get

$$\frac{HH_2(A/I^n)}{HH_2(A)} = \frac{HH_2^{(1)}(A/I^n)}{HH_2^{(1)}(A)} \oplus \frac{HH_2^{(2)}(A/I^n)}{HH_2^{(2)}(A)}.$$

But, for any  $k$ -algebra  $A$ , one has  $HH_2^{(1)}(A) = D_1^k(A)$ , and  $HH_2^{(2)}(A) = \Omega_{A/k}^2$  by [LO] (chapter 4), where  $D_*^k(A)$  denotes the Andre-Quillen homology of  $A$  over  $k$ . Moreover, for any ideal  $I \subset A$ , the map  $\Omega_{A/k}^2 \rightarrow \Omega_{(A/I)/k}^2$  is surjective. Thus, we get  $\frac{HH_2(A/I^n)}{HH_2(A)} = \frac{D_1^k(A/I^n)}{D_1^k(A)}$  and similarly for  $B$ . Now, from [LO] (chapter 3), we have a diagram of exact sequences

$$\begin{array}{ccccc} D_1^k(A) & \rightarrow & D_1^k(A/I^n) & \rightarrow & D_1^A(A/I^n) \\ \downarrow & & \downarrow & & \downarrow \\ D_1^k(B) & \rightarrow & D_1^k(B/I^n) & \rightarrow & D_1^B(B/I^n), \end{array}$$

and  $D_1^A(A/I^n) = I^n/I^{2n} = D_1^B(B/I^n)$ . This proves the required injectivity.  $\square$

**Lemma 2.2** *Let  $A$  be a regular ring which is essentially of finite type over a field  $K$  of char. zero. Let  $I \subset A$  be an invertible ideal. Then, for any subfield  $k \subset K$ , and any  $n \geq 0$ , the natural map*

$$\frac{HH_n^k(A/I^2)}{HH_n^k(A)} \rightarrow \frac{HH_n^k(A/I)}{HH_n^k(A)}$$

*is zero.*

**Proof.** Since Hochschild homology commutes with localisation, we can assume that  $R$  is a regular local ring and  $I = (t)$  is a principal ideal. Let  $A \mapsto D_*^k(A)$  denote the Andre-Quillen homology functor. Then, these homology groups are given by

$$D_*^k(A) := H_*(\mathbb{L}_{A/k}),$$

where  $\mathbb{L}_{A/k}$  denotes the cotangent complex of  $A$  over  $k$  ([LO]). We first claim that  $D_i^k(A) = 0$  for  $i > 0$ , and  $D_i^k(A/I) = 0$  for  $i > 1$ .

First, notice that since  $k$  is of char. zero,  $D_i^k(K)$  is the direct limit of  $D_i^k(L)$ , where  $L$  is a subfield of  $K$  and finitely generated over  $k$ . Moreover,  $L$  can be viewed as a finite extension of a purely transcendental extension (of finite degree) over  $k$ . But, the Andre-Quillen homology of finite extension vanishes in char. 0, and a purely transcendental extension of finite degree is a localisation of a polynomial ring over  $k$  for which the Andre-Quillen Homology again vanishes. Since Andre-Quillen homology commutes with direct limits ([Q]), we conclude that  $D_i^k(K) = 0$  for  $i > 0$ . Now, we use this fact and the exact sequence ([LO])

$$D_i^k(K) \otimes A \rightarrow D_i^k(A) \rightarrow D_i^K(A) \rightarrow D_{i-1}^k(K) \otimes A,$$

to see that it is enough to prove the claimed statement over  $K$ . However, since  $A$  is smooth over  $K$ , and  $I$  is a local complete intersection ideal in  $A$ , we have  $D_i^K(A) = 0$  for  $i > 0$  and  $D_i^K(A/I) = 0$  for  $i > 1$  by the results of Avramov and Halperin ([AH]).

Since  $A$  is smooth over  $K$ , there is an isomorphism  $HH_i^k(A) \cong \Omega_{A/k}^i$  for any subfield  $k \subset K$  ([C]). Furthermore, since  $D_i^k(A/I) = 0$  for  $i > 1$ , we have by [LR] (Theorem 3.1 and Proposition 3.2),

$$\frac{HH_n^k(A/I)}{HH_n^k(A)} \cong \oplus_{1 \leq 2j \leq n} H^{n-2j} \left( \frac{F_I^{n-j}(\Omega_{A/k}^*)}{F_I^{n-j+1}(\Omega_{A/k}^*)} \right), \quad (2.2)$$

where  $F_I \Omega_{A/k}^*$  is a filtration for the DeRham complex  $\Omega_{A/k}^*$  whose successive quotients are given by

$$\begin{aligned} \left( \frac{F_I^{n-j}(\Omega_{A/k}^*)}{F_I^{n-j+1}(\Omega_{A/k}^*)} \right)_{n-2j} &= \frac{I^j}{I^{j+1}} \otimes_A \Omega_{A/k}^{n-2j} \text{ and} \\ \left( \frac{F_I^{n-j}(\Omega_{A/k}^*)}{F_I^{n-j+1}(\Omega_{A/k}^*)} \right)_{n-2j+1} &= \frac{I^{j-1}}{I^j} \otimes_A \Omega_{A/k}^{n-2j+1} \end{aligned}$$

Note that since  $I$  is an invertible ideal, all its powers are also invertible, and hence 2.2 holds for all powers of  $I$ . Since the lemma is trivial for  $n = 0$ , we can assume that  $n$  is positive, and so is  $j$ . In this case, we see that the natural map  $\frac{(I^2)^j}{(I^2)^{j+1}} \otimes_A \Omega_{A/k}^{n-2j} \longrightarrow \frac{I^j}{I^{j+1}} \otimes_A \Omega_{A/k}^{n-2j}$  is zero, and hence by comparing 2.2 for  $I$  and  $I^2$ , we see that for  $1 \leq 2j \leq n$ , the map

$$H^{n-2j} \left( \frac{F_{I^2}^{n-j}(\Omega_{A/k}^*)}{F_{I^2}^{n-j+1}(\Omega_{A/k}^*)} \right) \longrightarrow H^{n-2j} \left( \frac{F_I^{n-j}(\Omega_{A/k}^*)}{F_I^{n-j+1}(\Omega_{A/k}^*)} \right)$$

is zero. Now, we use 2.2 again to finish the proof.  $\square$

Let  $k$  be a field of char. 0. For any ideal  $I$  of a  $k$ -algebra  $A$ , let  $\Omega_{(A,I^n)/k}^i$  denote the kernel of the natural surjection  $\Omega_{A/k}^i \rightarrow \Omega_{(A/I)/k}^i$ .

**Lemma 2.3** *Let  $A$  be a reduced  $k$ -algebra, and let  $B$  be the normalisation of  $A$ . Let  $I$  be a conducting ideal for the normalisation. For any  $i \geq 1$ , the map*

$$\frac{\Omega_{(B,I^{i+1})/k}^i}{\Omega_{(A,I^{i+1})/k}^i} \longrightarrow \frac{\Omega_{(B,I)/k}^i}{\Omega_{(A,I)/k}^i}$$

*is zero.*

**Proof.** We first observe from the universal property of the module of Kahler differentials that  $\Omega_{(A,I)/k}^i$  is the submodule of  $\Omega_{A/k}^i$ , generated by the exterior forms of the type  $a_0 da_1 \wedge \cdots \wedge da_i$ , where  $a_p \in A$  for all  $p$  and  $a_p \in I$  for some  $p$ . Let  $F\Omega_{(A,I)/k}^i$  denote the submodule of  $\Omega_{(A,I)/k}^i$  generated by the exterior forms of the type  $a_0 da_1 \wedge \cdots \wedge da_i$ , with  $a_p \in I$  for all  $p$ . Then, it is enough to show that

$$\text{Image}(\Omega_{(B,I^{i+1})/k}^i \rightarrow \Omega_{(B,I)/k}^i) \subset \text{Image}(F\Omega_{(A,I)/k}^i \rightarrow \Omega_{(B,I)/k}^i). \quad (2.3)$$

We prove this by induction on  $i$ .

For  $i = 1$ , let  $w = adb$  with  $a$  or  $b$  in  $I^2$ . If  $a \in I^2$ , then can assume  $a = a_1 a_2$  with  $a_p \in I$ . In that case, one gets  $a_1 a_2 db = a_1 (d(a_2 b) - b d(a_2))$ , which is clearly in  $F\Omega_{(A,I)/k}^1$ . If  $b \in I^2$ , one proceeds similarly. This proves  $i = 1$  case. Suppose now that 2.3 holds for all  $j \leq i - 1$  with  $i > 1$ . Put  $w = a_0 da_1 \wedge \cdots \wedge da_i$  with some  $a_p$  in  $I^{i+1}$ .

**Case 1.**  $p = 0$

Can assume  $a_0 = a_0^1 \cdots a_0^{i+1}$ . Then

$$\begin{aligned} a_0^1 \cdots a_0^{i+1} da_1 \wedge \cdots \wedge da_i &= (a_0^1 \cdots a_0^i da_1 \wedge \cdots \wedge da_{i-1})(a_0^{i+1} da_i) \\ &= (a_0^1 \cdots a_0^i da_1 \wedge \cdots \wedge da_{i-1})(d(a_0^{i+1} a_i) - a_i da_0^{i+1}) \\ &= (a_0^1 \cdots a_0^i da_1 \wedge \cdots \wedge da_{i-1} \wedge d(a_0^{i+1} a_i)) - \\ &\quad (a_0^1 \cdots a_0^i a_i da_1 \wedge \cdots \wedge da_{i-1} \wedge da_0^{i+1}). \end{aligned}$$

The induction hypothesis now applies.

**Case 1.**  $p > 0$ .

The proof is exactly along the lines of case 1.  $\square$

**Lemma 2.4** *Let  $A$  be a reduced ring which is essentially of finite type over a field  $K$  of char. 0, and let  $B$  be the smooth normalisation of  $A$ . Let  $I$  be a*

conducting ideal for the normalisation which is invertible in  $B$ . Let  $k \subset K$  be a subfield. Then, for any  $i \geq 1$ , the natural map

$$\frac{HH_i^k(B, I^n)}{HH_i^k(A, I^n)} \longrightarrow \frac{HH_i^k(B, I)}{HH_i^k(A, I)}$$

is zero for all sufficiently large  $n$ .

**Proof.** We shall in fact show that this holds for all  $n \geq (i+1)^2$ . Consider the exact sequence for relative Hochschild homology

$$0 \rightarrow \frac{HH_{i+1}^k(B/I^n)}{HH_{i+1}^k(B)} \rightarrow HH_i^k(B, I^n) \rightarrow \text{Ker}(HH_i^k(B) \rightarrow HH_i^k(B/I^n)) \rightarrow 0.$$

Since  $S$  is smooth, we have seen that  $HH_i^k(B) = HH_i^{k, (i)}(B) = \Omega_{B/k}^i$ , and hence from the naturality of Hodge decomposition on Hochschild homology, we have  $\text{Ker}(HH_i^k(B) \rightarrow HH_i^k(B/I^n)) \cong \text{Ker}(\Omega_{B/k}^i \rightarrow \Omega_{(B/I^n)/k}^i) = \Omega_{(B, I^n)/k}^i$ . Thus, we get a diagram of exact sequences

$$\begin{array}{ccccccc} 0 & \rightarrow & \frac{HH_{i+1}^k(A/I^n)}{HH_{i+1}^k(A)} & \rightarrow & HH_i^k(A, I^n) & \rightarrow & \text{Ker}(HH_i^k(A) \rightarrow HH_i^k(A/I^n)) \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & \frac{HH_{i+1}^k(B/I^n)}{HH_{i+1}^k(B)} & \rightarrow & HH_i^k(B, I^n) & \longrightarrow & \Omega_{(B, I^n)/k}^i \longrightarrow 0. \end{array}$$

Taking quotients, we get exact sequence

$$\frac{HH_{i+1}^k(B/I^n)}{HH_{i+1}^k(B) + HH_{i+1}^k(A/I^n)} \rightarrow \frac{HH_i^k(B, I^n)}{HH_i^k(A, I^n)} \rightarrow \frac{\Omega_{(B, I^n)/k}^i}{\text{Ker}(HH_i^k(A) \rightarrow HH_i^k(A/I^n))} \rightarrow 0.$$

Furthermore, since  $\Omega_{A/k}^i = HH_i^{k, (i)}(A)$  ([LO]), and similarly for other rings, we see from the naturality of Hodge decomposition that  $\frac{\Omega_{(B, I^n)/k}^i}{\text{Ker}(HH_i^k(A) \rightarrow HH_i^k(A/I^n))} \cong \frac{\Omega_{(B, I^n)/k}^i}{\Omega_{(A, I^n)/k}^i}$ . Comparing above exact sequence for  $n = 1, n = (i+1)$  and  $n = (i+1)^2$ , we get a diagram

$$\begin{array}{ccccc} \frac{HH_{i+1}^k(B/I^{(i+1)^2})}{HH_{i+1}^k(B) + HH_{i+1}^k(A/I^{(i+1)^2})} & \rightarrow & \frac{HH_i^k(B, I^{(i+1)^2})}{HH_i^k(A, I^{(i+1)^2})} & \rightarrow & \frac{\Omega_{(B, I^{(i+1)^2})/k}^i}{\Omega_{(A, I^{(i+1)^2})/k}^i} \rightarrow 0 \\ \downarrow & & \downarrow & & \downarrow \\ \frac{HH_{i+1}^k(B/I^{i+1})}{HH_{i+1}^k(B) + HH_{i+1}^k(A/I^{i+1})} & \rightarrow & \frac{HH_i^k(B, I^{i+1})}{HH_i^k(A, I^{i+1})} & \rightarrow & \frac{\Omega_{(B, I^{i+1})/k}^i}{\Omega_{(A, I^{i+1})/k}^i} \rightarrow 0 \\ \downarrow & & \downarrow & & \downarrow \\ \frac{HH_{i+1}^k(B/I)}{HH_{i+1}^k(B) + HH_{i+1}^k(A/I)} & \rightarrow & \frac{HH_i^k(B, I)}{HH_i^k(A, I)} & \rightarrow & \frac{\Omega_{(B, I)/k}^i}{\Omega_{(A, I)/k}^i} \rightarrow 0 \end{array}$$



The two vertical maps on the left are zero by lemma 2.2, and the two vertical maps on the right are zero by lemma 2.3. A diagram chase shows that the composite map in the middle is zero.  $\square$

**Lemma 2.5** *Under the conditions of lemma 2.4, the map*

$$HH_1^k(A, I^n) \longrightarrow HH_1^k(B, I^n)$$

*is injective for all sufficiently large  $n$ .*

**Proof.** For any subring  $R \hookrightarrow A$ , let  $D_*^R(A, I)$  be the relative Andre-Quillen homology defined as the homology groups of the complex  $\text{Ker}(\mathbb{L}_{A/R} \rightarrow \mathbb{L}_{(A/I)/R})$ . These groups fit into long exact sequence of relative Andre-Quillen homology. As in [LO], there are natural maps  $D_i^R(A, I) \rightarrow HH_{i+1}^{R, (1)}(A, I)$ . Comparing these groups using long exact sequences of Andre-Quillen homology and Hochschild homology, and using the isomorphism  $D_i^R(A) \cong HH_{i+1}^{R, (1)}(A)$ , one gets isomorphism  $D_i^R(A, I) \cong HH_{i+1}^{R, (1)}(A, I)$  for all  $i \geq 0$ . Thus, we need to show that the natural map  $D_0^k(A, I^n) \rightarrow D_0^k(B, I^n)$  is injective for all large  $n$ . Using the base change long exact sequence of Andre-Quillen homology ([LO]), one gets exact sequence

$$\Omega_{K/k}^1 \otimes I^n \rightarrow D_0^k(A, I^n) \rightarrow D_0^K(A, I^n) \rightarrow 0$$

Comparing this exact sequence for  $A$  and  $B$ , we have a commutative diagram

$$\begin{array}{ccccccc} \Omega_{K/k}^1 \otimes I^n & \rightarrow & D_0^k(A, I^n) & \rightarrow & D_0^K(A, I^n) & \rightarrow & 0 \\ \parallel & & \downarrow & & \downarrow & & \\ \Omega_{K/k}^1 \otimes I^n & \rightarrow & D_0^k(B, I^n) & \rightarrow & D_0^K(B, I^n) & \rightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ 0 \rightarrow \Omega_{K/k}^1 \otimes B & \longrightarrow & \Omega_{B/k}^1 & \longrightarrow & \Omega_{B/K}^1 & \longrightarrow & 0, \end{array}$$

in which all the rows are exact, and the second diagram is a part of long exact sequence of relative Andre-Quillen homology of  $B$  and  $I^n$ , and using the isomorphism  $D_0^k(B) \cong \Omega_{B/k}^1$ . The bottom sequence is exact on the left since  $B$  is smooth over  $K$ . A diagram chase now shows that all the rows are exact on the left. Now, lemma 2.1 and Snake lemma complete the proof.  $\square$

**Lemma 2.6** *Under the conditions of lemma 2.4, the natural map*

$$HH_1^k(A, B, I^n) \longrightarrow HH_1^k(A, B, I)$$

of double relative Hochschild homology groups is zero for all sufficiently large  $n$ .

**Proof.** The long exact sequence of relative Hochschild homology gives exact sequence

$$0 \rightarrow \frac{HH_2^k(B, I^n)}{HH_2^k(A, I^n)} \rightarrow HH_1^k(A, B, I^n) \rightarrow \text{Ker}(HH_1^k(A, I^n) \rightarrow HH_1^k(B, I^n)) \rightarrow 0.$$

But the last group vanishes by lemma 2.5. Furthermore, the map  $\frac{HH_2^k(B, I^n)}{HH_2^k(A, I^n)} \rightarrow \frac{HH_2^k(B, I)}{HH_2^k(A, I)}$  is zero for all large  $n$  by lemma 2.4.  $\square$

**Corollary 2.7** *Under the conditions of lemma 2.4, the natural map*

$$HC_1^k(A, B, I^n) \longrightarrow HC_1^k(A, B, I)$$

*of double relative cyclic homology groups is zero for all sufficiently large  $n$ .*

**Proof.** In view of the above lemma, it's enough to show that the natural map  $HH_1^k(A, B, I^n) \longrightarrow HC_1^k(A, B, I^n)$  is surjective for all  $n$ . But, the SBI sequence ([LO]) of double relative Hochschild and cyclic homology groups gives exact sequence

$$HH_1^k(A, B, I^n) \longrightarrow HC_1^k(A, B, I^n) \xrightarrow{S} HC_{-1}^k(A, B, I^n).$$

Another exact sequence of relative cyclic homology gives exact sequence

$$HC_0^k(A, I^n) \longrightarrow HC_0^k(B, I^n) \longrightarrow HC_{-1}^k(A, B, I^n) \longrightarrow HC_{-1}^k(A, I^n).$$

However,  $HC_0^k(A, I^n) \cong I^n \cong HC_0^k(B, I^n)$ , and  $HC_{-1}^k(A, I^n) = 0$ . This finishes the proof.  $\square$

**Remark.** We point out here that it is already known that the map  $HC_0^k(A, B, I^2) \rightarrow HC_0^k(A, B, I)$  is zero ([CGW]).

We conclude this section with the following two lemmas.

**Lemma 2.8** *Let  $A$  be an one-dimensional local domain, essentially of finite type over an algebraically closed field of char. 0, let  $B$  be the normalisation of  $A$ . Let  $k \subset K$  be any subfield. Then the natural map*

$$\text{Ker}(HH_1^k(A) \rightarrow HH_1^k(B)) \longrightarrow \text{Ker}(HH_1^K(A) \rightarrow HH_1^K(B))$$

*is an isomorphism.*

**Proof.** We first observe that  $HH_1^k(A) = \Omega_{A/k}^1$  and similarly for  $B$ . Thus, we can work with Kahler differentials. The above map is already injective, so we need to show only surjectivity. Note that since  $B$  is regular,  $\Omega_{B/k}^1$  is a free  $B$ -module (not necessarily finitely generated). Thus, the map  $\Omega_{K/k}^1 \otimes_K A \hookrightarrow \Omega_{K/k}^1 \otimes_K B \rightarrow \Omega_{B/k}^1$  is injective. Furthermore, since  $K$  is algebraically closed, the map  $\Omega_{K/k}^1 \rightarrow \Omega_{A/k}^1 \rightarrow \Omega_{K/k}^1$  is naturally split. In particular,  $\Omega_{K/k}^1$  is naturally a direct summand of  $\Omega_{A/k}^1$ .

Put  $\overline{\Omega_{A/k}^1} = \text{Ker}(\Omega_{A/k}^1 \rightarrow \Omega_{B/k}^1)$ . We define  $\overline{\Omega_{A/K}^1}$  similarly. Let  $F$  denote the field of fractions of  $A$ . Then it is easy to see that  $\overline{\Omega_{A/k}^1} = \text{Tor}_A^1(F/A, \Omega_{A/k}^1)$ , and one has similar interpretation for  $\overline{\Omega_{A/K}^1}$ . This follows because the map  $\Omega_{B/k}^1 \rightarrow \Omega_{F/k}^1$  is injective. Thus, we need to show that the map  $\text{Tor}_A^1(F/A, \Omega_{A/k}^1) \rightarrow \text{Tor}_A^1(F/A, \Omega_{A/K}^1)$  is surjective. But, using the exact sequence

$$0 \rightarrow \Omega_{K/k}^1 \otimes_K A \rightarrow \Omega_{A/k}^1 \rightarrow \Omega_{A/K}^1 \rightarrow 0,$$

one gets a long exact sequence

$$\text{Tor}_A^1(F/A, \Omega_{A/k}^1) \rightarrow \text{Tor}_A^1(F/A, \Omega_{A/K}^1) \rightarrow \Omega_{K/k}^1 \otimes_K F/A \xrightarrow{\phi} \Omega_{A/k}^1 \otimes_A F/A.$$

Hence, it is enough to show that  $\phi$  is injective. However, one has a factorisation  $\Omega_{K/k}^1 \otimes_K F/A \rightarrow \Omega_{A/k}^1 \otimes_A F/A \rightarrow A \otimes_K (\Omega_{A/k}^1 \otimes_A F/A)$ . Thus, it is enough to show that the composite map is injective. However,

$$\begin{aligned} \Omega_{K/k}^1 \otimes_K F/A &= (\Omega_{K/k}^1 \otimes_K A) \otimes_A F/A \hookrightarrow (\Omega_{A/k}^1 \otimes_K A) \otimes_A F/A \\ &\cong A \otimes_K (\Omega_{A/k}^1 \otimes_A F/A). \end{aligned}$$

Here, the injective arrow follows because  $\Omega_{K/k}^1$  is naturally a direct summand of  $\Omega_{A/k}^1$  as observed before. This proves the desired injectivity.  $\square$

**Lemma 2.9** *Let  $A$  and  $B$  be as in lemma 2.8. Let  $\mathfrak{m}$  and  $\mathfrak{M}$  denote the Jacobson radicals of  $A$  and  $B$  respectively. Then the natural map*

$$\text{Ker}(HH_1^k(A, \mathfrak{m}) \rightarrow HH_1^k(B, \mathfrak{M})) \rightarrow \text{Ker}(HC_1^k(A, \mathfrak{m}) \rightarrow HC_1^k(B, \mathfrak{M})).$$

*is injective.*

**Proof.** We consider the following diagram of exact sequences coming from the SBI-sequence

$$\begin{array}{ccccccc} 0 & \rightarrow & HC_0^k(A, \mathfrak{m}) & \rightarrow & HH_1^k(A, \mathfrak{m}) & \rightarrow & HC_1^k(A, \mathfrak{m}) \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & HC_0^k(B, \mathfrak{M}) & \rightarrow & HH_1^k(B, \mathfrak{M}) & \rightarrow & HC_1^k(B, \mathfrak{M}) \rightarrow 0, \end{array}$$

where the first arrow from left in the bottom sequence is injective because the composite map  $\mathfrak{M} = HC_0^k(B, \mathfrak{M}) \longrightarrow HH_1^k(B, \mathfrak{M}) \longrightarrow \Omega_{B/k}^1$  is injective as  $B$  is regular. Also,  $HC_0^k(A, \mathfrak{m}) \cong \mathfrak{m}$ , again using the long exact sequence for relative cyclic homology. Thus the left-most vertical map is just the inclusion  $\mathfrak{m} \hookrightarrow \mathfrak{M}$ . A diagram chase now proves the lemma.  $\square$

### 3 Mayer-Vietoris sequences in $K$ -theory and Cyclic homology

Our goal in this section is to establish some Mayer-Vietoris type exact sequences in  $K$ -theory and cyclic homology. These sequences will be one of our main tools to prove main theorems. Let  $A$  be an one-dimensional reduced local ring, which is essentially of finite type over an algebraically closed field  $K$  of characteristic zero. Let  $\mathfrak{m}$  denote the Jacobson radical of  $A$ . Let  $B$  be the normalisation of  $A$  with the Jacobson radical  $\mathfrak{M}$ . Note that  $B$  is a direct product of regular semi-local domains. Then, for any radical ideal  $I$  of  $B$ , one has a fibration of  $K$ -theory spectra

$$K(B, I) \longrightarrow K(B, \mathfrak{M}) \longrightarrow K(B/I, \mathfrak{M}/I).$$

**Lemma 3.1** *Let  $A$  be a reduced local ring as above with the maximal ideal  $\mathfrak{m}$ , and let  $B$  be the normalisation of  $A$  with Jacobson radical  $\mathfrak{M}$ . Then for any conducting ideal  $I$ , one has ‘Mayer-Vietoris’ exact sequences*

$$K_2(A, \mathfrak{m}) \longrightarrow K_2(B, \mathfrak{M}) \oplus K_2(A/I, \mathfrak{m}/I) \longrightarrow K_2(B/I, \mathfrak{M}/I) \longrightarrow 0.$$

$$HC_1(A, \mathfrak{m}) \longrightarrow HC_1(B, \mathfrak{M}) \oplus HC_1(A/I, \mathfrak{m}/I) \longrightarrow HC_1(B/I, \mathfrak{M}/I) \longrightarrow 0.$$

**Proof.** From the above fibration of  $K$ -theory spectra, one has diagrams of exact sequences

$$\begin{array}{ccccccc} & & K_2(A, I) & \longrightarrow & K_2(A, \mathfrak{m}) & \longrightarrow & K_2(A/I, \mathfrak{m}/I) \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ K_3(B/I, \mathfrak{M}/I) & \longrightarrow & K_2(B, I) & \longrightarrow & K_2(B, \mathfrak{M}) & \longrightarrow & K_2(B/I, \mathfrak{M}/I) \rightarrow 0 \\ & \searrow & \downarrow & & & & \\ & & K_1(A, B, I) & & & & \end{array}$$

$$\begin{array}{ccccccc}
& HC_1(A, I) & \longrightarrow & HC_1(A, \mathfrak{m}) & \longrightarrow & HC_1(A/I, \mathfrak{m}/I) & \longrightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
HC_2(B/I, \mathfrak{M}/I) & \longrightarrow & HC_1(B, I) & \longrightarrow & HC_1(B, \mathfrak{M}) & \longrightarrow & HC_1(B/I, \mathfrak{M}/I) \longrightarrow 0 \\
& \searrow & \downarrow & & & & \\
& & HC_0(A, B, I) & & & & 
\end{array}$$

Diagram 2

The surjectivity of the last horizontal map in the first diagram follows since the map  $B/I \longrightarrow B/\mathfrak{M}$  is split surjective, and then compare long exact  $K$ -theory sequence for pairs  $(B, \mathfrak{M})$  and  $(B/I, \mathfrak{M}/I)$ . Similar argument holds in the second diagram. This also proves the surjectivity of the last maps in the lemma. Now a diagram chase shows that it is enough to prove that the slanted arrows in both diagrams are surjective. However, we know that by [GO] and [CO], there are isomorphisms

$$K_3(B/I, \mathfrak{M}/I) \cong HC_2(B/I, \mathfrak{M}/I), \text{ and } K_1(A, B, I) \cong HC_0(A, B, I).$$

Here, all Hochschild and Cyclic homologies are taken with respect to the field of rational numbers  $\mathbb{Q}$ . Furthermore, one has a commutative diagram

$$\begin{array}{ccc}
HH_2(B/I, \mathfrak{M}/I) & \longrightarrow & HH_0(A, B, I) \\
\downarrow & & \parallel \\
HC_2(B/I, \mathfrak{M}/I) & \longrightarrow & HC_0(A, B, I),
\end{array}$$

where the right equality follows from the SBI-sequence of double relative Hochschild and Cyclic homology ([LO]) or by direct computation. But, from the proof of Theorem 1.2 of [CGW], the map  $HH_2(B/I, \mathfrak{M}/I) \longrightarrow HH_0(A, B, I)$  is surjective. This proves desired surjectivity of both the slanted arrows.  $\square$

**Corollary 3.2** *With notations as in the above lemma, the maps*

$$\begin{aligned}
& \text{Ker}(K_2(A, \mathfrak{m}) \rightarrow K_2(B, \mathfrak{M})) \longrightarrow \text{Ker}(K_2(A/I, \mathfrak{m}/I) \rightarrow K_2(B/I, \mathfrak{M}/I)), \\
& \text{Ker}(HC_1(A, \mathfrak{m}) \rightarrow HC_1(B, \mathfrak{M})) \longrightarrow \text{Ker}(HC_1(A/I, \mathfrak{m}/I) \rightarrow HC_1(B/I, \mathfrak{M}/I))
\end{aligned}$$

*are surjective.*

**Proof.** Follows directly from the ‘Mayer-Vietoris’ sequence of the lemma.  $\square$

**Remark.** The second part of the corollary was also established in [CGW].

**Lemma 3.3** *Under the hypothesis of lemma 3.1, the map*

$$\frac{K_3(B/I^2, \mathfrak{M}/I^2)}{K_3(B, \mathfrak{M})} \longrightarrow \frac{K_3(B/I, \mathfrak{M}/I)}{K_3(B, \mathfrak{M})}$$

*is zero.*

**Proof.** Note that  $K_3(B/I, \mathfrak{M}/I) = K_3^{\text{nil}}(B/I, \mathfrak{M}/I)$ , and the latter is a  $\mathbb{Q}$ -vector space. Hence, both groups above remain unchanged even after we mod out torsion part of  $K_3(B, \mathfrak{M})$ . Using Adam's operations on rational relative  $K$ -theory as in [L] (see also [C]), one has eigenspace decomposition

$$K_3(B/I^n, \mathfrak{M}/I^n) = K_3^{(2)}(B/I^n, \mathfrak{M}/I^n) \oplus K_3^{(3)}(B/I^n, \mathfrak{M}/I^n).$$

Further,  $K_3^{(3)}(B/I^n, \mathfrak{M}/I^n) = K_3^M(B/I^n, \mathfrak{M}/I^n)$ , where the latter is the relative Milnor  $K$ -group as defined by Levine ([L]). By naturality of eigenspace decomposition, one gets

$$\frac{K_3(B/I^n, \mathfrak{M}/I^n)}{K_3(B, \mathfrak{M})} \cong \frac{K_3^{(2)}(B/I^n, \mathfrak{M}/I^n)}{K_3^{(2)}(B, \mathfrak{M})} \oplus \frac{K_3^M(B/I^n, \mathfrak{M}/I^n)}{K_3^{(3)}(B, \mathfrak{M})}.$$

Now,  $B$  is a direct product of regular semi-local domains in which all height 1 prime ideals are principal, and since  $\mathfrak{M}$  is the product of all maximal ideals, we see the pair  $(B, \mathfrak{M})$  satisfies the MV-Property of Levine. We

**Claim.** There is a surjection

$$K_3^M(B, \mathfrak{M}) \twoheadrightarrow \text{Ker}(K_3^M(B) \twoheadrightarrow K_3^M(B/\mathfrak{M})).$$

For this, we use the eigen pieces of the long exact rel.  $K$ -theory sequence, to get exact sequence

$$K_3^{(3)}(B, \mathfrak{M}) \longrightarrow \text{Ker}(K_3^{(3)}(B) \longrightarrow K_3^{(3)}(B/\mathfrak{M})).$$

But all these are corresponding Milnor  $K$ -groups by [L]. This proves the claim. We point out here that the isomorphism  $K_3^M(B, \mathfrak{M}) \cong F^3 K_3(B, \mathfrak{M}) \cong K_3^{(3)}(B, \mathfrak{M})$  is known only after we mod out two torsion elements. But as remarked in the beginning of the proof of the lemma, this does not affect the statement of the lemma.

Now, using this claim and the the fact that the surjection  $B/I^n \twoheadrightarrow B/\mathfrak{M}$  splits, one has a diagram of exact sequences

$$\begin{array}{ccccccc} K_3^M(B, \mathfrak{M}) & \longrightarrow & K_3^M(B) & \longrightarrow & K_3^M(B/\mathfrak{M}) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ 0 \longrightarrow & K_3^M(B/I^n, \mathfrak{M}/I^n) & \longrightarrow & K_3^M(B/I^n) & \longrightarrow & K_3^M(B/\mathfrak{M}) & \longrightarrow 0, \end{array}$$

which in turn gives a surjection

$$K_3^M(B, \mathfrak{M}) \twoheadrightarrow K_3^M(B/I^n, \mathfrak{M}/I^n).$$

Applying this in the eigenspace decomposition above, we obtain

$$\begin{aligned} \frac{K_3(B/I^n, \mathfrak{M}/I^n)}{K_3(B, \mathfrak{M})} &= \frac{K_3^{(2)}(B/I^n, \mathfrak{M}/I^n)}{K_3^{(2)}(B, \mathfrak{M})} \\ &= \frac{HC_2^{(1)}(B/I^n, \mathfrak{M}/I^n)}{K_3^{(2)}(B, \mathfrak{M})} \\ &= \frac{\prod_i HC_2^{(1)}(B\mathfrak{M}_i/I^n, \mathfrak{M}_i/I^n)}{K_3^{(2)}(B, \mathfrak{M})} \\ &= \frac{\prod_i D_1^{\mathbb{Q}}(\mathbb{Q}[t^i/t_i^{r_i}, (t_i)] \otimes HH_0^{\mathbb{Q}}(k))}{K_3^{(2)}(B, \mathfrak{M})}, \end{aligned}$$

where the sum is taken over maximal ideals of  $B$ . Here,  $D^{\mathbb{Q}}(A)$  denotes the Andre-Quillen homology of a  $\mathbb{Q}$ -algebra  $A$  ([LO]) and the last equality follows from the computation of the Cyclic homology of truncated polynomial algebras in ([LO], sec. 4.6). Now, the proof of the lemma follows from the following

**Sublemma 3.4** *Let  $B_r$  denote the truncated polynomial ring  $\mathbb{Q}[t]/(t^r)$ . Then, the map  $D_1^{\mathbb{Q}}(B_{2r}) \longrightarrow D_1^{\mathbb{Q}}(B_r)$  is zero.*

**Proof.** Since we are dealing with rational coefficients, we shall ignore the index  $\mathbb{Q}$  in this proof. Note that  $D_1(B_r) = \frac{D_1(B_r)}{D_1(\mathbb{Q}[t])} (\mathbb{Q}[t] \text{ is smooth over } \mathbb{Q})$ , which in turn is same as  $\frac{HH_2^{(1)}(B_r)}{HH_2^{(1)}(\mathbb{Q}[t])}$  by [LO]. But the map

$$\frac{HH_2(B_{2r})}{HH_2(\mathbb{Q}[t])} \longrightarrow \frac{HH_2(B_r)}{HH_2(\mathbb{Q}[t])}$$

is zero by lemma 2.2. □

**Corollary 3.5** *Let  $F_2(B, I)$  denote  $\text{Ker}(K_2(B, I) \longrightarrow K_2(B, \mathfrak{M}))$ . Then the map  $F_2(B, I^2) \longrightarrow F_2(B, I)$  is zero.*

**Proof.** This follows once we observe that

$$F_2(B, I) = \frac{K_3(B/I, \mathfrak{M}/I)}{K_3(B, \mathfrak{M})},$$

using the  $K$ -theory long exact sequence for the map of pairs  $(B, I) \longrightarrow (B, \mathfrak{M})$ , and then using above lemma. □

The following is our main result of this section, which is a stronger version of lemma 3.1.

**Theorem 3.6** *Consider the hypothesis of lemma 3.1. Then, there exists a conducting ideal  $I$  such that one has ‘Mayer-Vietoris’ exact sequences*

$$0 \rightarrow K_2(A, \mathfrak{m}) \rightarrow K_2(B, \mathfrak{M}) \oplus K_2(A/I, \mathfrak{m}/I) \rightarrow K_2(B/I, \mathfrak{M}/I) \rightarrow 0. \quad (3.4)$$

$$0 \rightarrow HC_1(A, \mathfrak{m}) \rightarrow HC_1(B, \mathfrak{M}) \oplus HC_1(A/I, \mathfrak{m}/I) \rightarrow HC_1(B/I, \mathfrak{M}/I) \rightarrow 0. \quad (3.5)$$

**Proof.** In view of lemma 3.1, we only need to prove injectivity of first maps in both sequences for some conducting ideal  $I$ . We shall in fact show that given a conducting ideal  $I$ , this holds for all sufficiently large powers of  $I$ . We fix some notations before beginning the proof. For any conducting ideal  $I$ , let

$$F(I) := \text{Ker}(K_2(A, \mathfrak{m}) \rightarrow K_2(A/I, \mathfrak{m}/I)) \bigcap \text{Ker}(K_2(A, \mathfrak{m}) \rightarrow K_2(B, \mathfrak{M})),$$

$$F(A) := \text{Ker}(K_2(A, \mathfrak{m}) \rightarrow K_2(B, \mathfrak{M})), \text{ and}$$

$$F(A/I) := \text{Ker}(K_2(A/I, \mathfrak{m}/I) \rightarrow K_2(B/I, \mathfrak{M}/I)).$$

Then, lemma 3.1 implies that one has a short exact sequence

$$0 \longrightarrow F(I) \longrightarrow F(A) \longrightarrow F(A/I) \longrightarrow 0. \quad (3.6)$$

We consider the diagram of exact sequences

$$\begin{array}{ccccccc} & & & & K_2(A, B, I) & & \\ & & & & \downarrow & & \\ 0 & \longrightarrow & F_2(A, I) & \longrightarrow & K_2(A, I) & \longrightarrow & K_2(A, \mathfrak{m}) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & F_2(B, I) & \longrightarrow & K_2(B, I) & \longrightarrow & K_2(B, \mathfrak{M}) \\ & & & \searrow & \downarrow & & \\ & & & & K_1(A, B, I) & & \end{array}$$

where the groups on the left are as defined in corollary 3.5. First we claim that the slanted arrow in this diagram is surjective. But this follows directly once we chase diagram 2 and observe in the proof of lemma 3.1 that the slanted arrow in that diagram is surjective. Thus, a diagram chase above gives exact sequence

$$K_2(A, B, I) \longrightarrow F(I) \longrightarrow \frac{F_2(B, I)}{F_2(A, I)} \xrightarrow{\beta_I} K_1(A, B, I) \rightarrow 0$$



However, we have natural isomorphism  $K_2(A, B, I) \cong HC_1(A, B, I)$  by Cortinas' theorem ([CO]). Using this in this exact sequence, and comparing the resulting sequences for various powers of  $I$ , we get a diagram

$$\begin{array}{ccccc} HC_1(A, B, I^n) & \longrightarrow & F(I^n) & \longrightarrow & \frac{F_2(B, I^n)}{F_2(A, I^n)} \\ \downarrow & & \downarrow & & \downarrow \\ HC_1(A, B, I) & \longrightarrow & F(I) & \longrightarrow & \frac{F_2(B, I)}{F_2(A, I)} \end{array}$$

By corollary 3.5, the right vertical map is zero for  $n \geq 2$ , and the left vertical map is zero for  $n \gg 0$  by corollary 2.7. Let  $N$  be an integer such that both these maps are zero. Now, we repeat the same argument in the above diagram with  $I$  replaced by  $J = I^N$  to get a diagram

$$\begin{array}{ccccc} HC_1(A, B, J^n) & \longrightarrow & F(J^n) & \longrightarrow & \frac{F_2(B, J^n)}{F_2(A, J^n)} \\ \downarrow & & \downarrow & & \downarrow \\ HC_1(A, B, J) & \longrightarrow & F(J) & \longrightarrow & \frac{F_2(B, J)}{F_2(A, J)} \\ \downarrow & & \downarrow & & \downarrow \\ HC_1(A, B, I) & \longrightarrow & F(I) & \longrightarrow & \frac{F_2(B, I)}{F_2(A, I)}, \end{array}$$

such that all the vertical maps on the left and right ends are zero. A diagram chase above now shows that for all  $n \gg 0$ , the map  $F(I^n) \rightarrow F(I)$  is zero. Applying this in 3.6, we see that  $F(A) \rightarrow F(A/I^n)$  is an isomorphism for all large powers of  $I$ . This proves the exactness of first sequence. The case of cyclic homology follows along exactly similar lines. In fact, we have reduced  $K$ -theory problem to Cyclic homology problem in the above proof.  $\square$

**Corollary 3.7** *Let  $(A, \mathfrak{m})$  be a reduced one-dimensional local ring, and let  $B$  be a reduced semi-local ring containing  $A$  and contained in the normalisation of  $A$ . Let  $\mathfrak{M}$  be the Jacobson radical of  $B$ . Then there is a conducting ideal  $I \subset B$  such that one has Mayer-Vietoris exact sequences as in Theorem 3.6.*

**Proof.** To prove the injectivity of the map  $K_2(A, \mathfrak{m}) \rightarrow K_2(B, \mathfrak{M}) \oplus K_2(A/I, \mathfrak{m}/I)$ , observe that we can choose a  $I$  to be a conducting ideal for the normalisation of  $A$  and hence it will also be conducting ideal for  $A \rightarrow B$ . This reduces the proof to the case when  $B$  is the normalisation of  $A$ . To prove the exactness of the sequence

$$K_2(A, \mathfrak{m}) \rightarrow K_2(B, \mathfrak{M}) \oplus K_2(A/I, \mathfrak{m}/I) \rightarrow K_2(B/I, \mathfrak{M}/I) \rightarrow 0,$$

we use the exact sequence (which always holds) from the Diagram 3

$$K_1(A, B, I) \longrightarrow \frac{K_2(B, \mathfrak{M})}{K_2(A, \mathfrak{m})} \longrightarrow \frac{K_2(B/I, \mathfrak{M}/I)}{K_2(A/I, \mathfrak{m}/I)} \longrightarrow 0$$

and use the Cortinas' isomorphism  $K_1(A, B, I) \cong HC_0(A, B, I) \cong I/I^2 \otimes \Omega_{B/A}^1$ , which holds even if  $B$  is not normal. Now, we compare this exact sequence for  $I$  and  $I^2$  and argue as before to finish the proof. The case of Cyclic homology is along the similar lines.  $\square$

## 4 Proofs of main theorems

**Proof of Theorem 1.1.** Let  $A$  be an one-dimensional local domain, essentially of finite type over an algebraically closed field  $K$  of characteristic 0. Let  $\mathfrak{m}$  denote the Jacobson radical of  $A$ . Let  $B$  be the normalisation of  $A$  with Jacobson radical  $\mathfrak{M}$ . Let  $F$  be the field of fractions of  $A$ . Assume that the ‘Artinian Geller Conjecture’ holds, and  $A$  is singular. Since  $B$  is a regular semi-local domain, the map  $K_2(B) \longrightarrow K_2(F)$  is injective by Quillen’s proof of Gersten conjecture. Hence it suffices to prove that the map  $K_2(A) \longrightarrow K_2(B)$  is not injective. Consider the commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \rightarrow & K_2(A, \mathfrak{m}) & \rightarrow & K_2(A) & \rightarrow & K_2(A/\mathfrak{m}) \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & K_2(B, \mathfrak{M}) & \rightarrow & K_2(A) & \rightarrow & K_2(B/\mathfrak{M}) \rightarrow 0. \end{array}$$

Diagram 3

Here, the map  $K_2(A, \mathfrak{m}) \longrightarrow K_2(A)$  is injective since  $K \longrightarrow A \longrightarrow A/\mathfrak{m} = K$  is split, as  $K$  is algebraically closed. By the same reason, the right-most vertical map is injective, since  $B/\mathfrak{M}$  is some copies of  $K$ . A diagram chase shows that it is enough to show that the left-most vertical map is not injective. We choose a conducting ideal  $I$  for the normalisation such that  $I \subset \mathfrak{m}^2$  and moreover, one has the Mayer-Vietoris exact sequences as in Theorem 3.6. Thus, we have

$$\text{Ker}(K_2(A, \mathfrak{m}) \rightarrow K_2(B, \mathfrak{M})) \cong \text{Ker}(K_2(A/I, \mathfrak{m}/I) \rightarrow K_2(B/I, \mathfrak{M}/I)). \quad (4.7)$$

Using the same diagram as above with  $A$  (resp.  $B$ ) replaced with  $A/I$  (resp.  $B/I$ ), and observing that  $K_3(B/I) \rightarrow K_3(B/\mathfrak{M})$ , we see that

$$\text{Ker}(K_2(A/I) \rightarrow K_2(B/I)) = \text{Ker}(K_2(A/I, \mathfrak{m}/I) \rightarrow K_2(B/I, \mathfrak{M}/I)). \quad (4.8)$$

Now, if  $A$  is singular, then  $\mathfrak{m}$  is not a principal ideal, and since  $I \subset \mathfrak{m}^2$ , Nakayama's Lemma implies that  $\mathfrak{m}/I$  is also not a principal ideal in  $A/I$ . In particular,  $A/I$  is not a principal ideal algebra though it is a subalgebra of  $B/I$ , which is a principal ideal algebra. Hence by 'Artinian Geller Conjecture',  $\text{Ker}(K_2(A/I) \rightarrow K_2(B/I)) \neq 0$ . Now, we use 4.8 and then 4.7 to finish the proof.  $\square$

**Proof of Theorem 1.2.** We first observe that for any subfield  $k \subset K$ ,  $HH_2^k(B/\mathfrak{M}) = HH_2^{k,(1)}(B/\mathfrak{M}) \oplus HH_2^{k,(2)}(B/\mathfrak{M})$  by the Hodge decomposition on Hochschild homology. But  $HH_2^{k,(1)}(B/\mathfrak{M}) = D_1^k(B/\mathfrak{M}) = 0$  ([C]). Also,  $HH_2^{k,(2)}(B/\mathfrak{M}) = \Omega_{(B/\mathfrak{M})/k}^2$ , and  $\Omega_{B/k}^2 \twoheadrightarrow \Omega_{(B/\mathfrak{M})/k}^2$ . In particular,  $HH_2^k(B) \twoheadrightarrow HH_2^k(B/\mathfrak{M})$ . Thus, using the long exact sequence for relative Hochschild homology, we see that

$$\begin{aligned} \text{Ker}(\Omega_{A/k}^1 \longrightarrow \Omega_{B/k}^1) &= \text{Ker}(HH_1^k(A) \longrightarrow HH_1^k(B)) \cong \\ &\text{Ker}(HH_1^k(A, \mathfrak{m}) \longrightarrow HH_1^k(B, \mathfrak{M})). \end{aligned}$$

Now, suppose that Berger's conjecture holds, and  $A$  is singular. Then the map  $\Omega_{A/K}^1 \longrightarrow \Omega_{B/K}^1$  is not injective. Hence, by lemma 2.8, the map  $HH_1^{\mathbb{Q}}(A) \rightarrow HH_1^{\mathbb{Q}}(B)$  is not injective, and the above isomorphism implies that the map  $HH_1^{\mathbb{Q}}(A, \mathfrak{m}) \rightarrow HH_1^{\mathbb{Q}}(B, \mathfrak{M})$  is not injective. Now, we use lemma 2.9 to conclude that

$$\text{Ker}(HC_1^{\mathbb{Q}}(A, \mathfrak{m}) \longrightarrow HC_1^{\mathbb{Q}}(B, \mathfrak{M})) \neq 0. \quad (4.9)$$

We choose a conducting ideal  $I$  for the normalisation of  $A$  so that we have Mayer-Vietoris exact sequences as in Theorem 3.6. Then we get an isomorphism as in 4.7 and also an isomorphism

$$\text{Ker}(HC_1^{\mathbb{Q}}(A, \mathfrak{m}) \rightarrow HC_1^{\mathbb{Q}}(B, \mathfrak{M})) \cong \text{Ker}(HC_1^{\mathbb{Q}}(A/I, \mathfrak{m}/I) \rightarrow HC_1^{\mathbb{Q}}(B/I, \mathfrak{M}/I)). \quad (4.10)$$

However, by Goodwillie's theorem ([GO]), the maps

$$K_2(A/I, \mathfrak{m}/I) \longrightarrow HC_1^{\mathbb{Q}}(A/I, \mathfrak{m}/I), \text{ and}$$

$$K_2(B/I, \mathfrak{M}/I) \longrightarrow HC_1^{\mathbb{Q}}(B/I, \mathfrak{M}/I)$$

are isomorphisms. Now, we combine 4.10, 4.8 and 4.7 to conclude that  $\text{Ker}(K_2(A, \mathfrak{m}) \rightarrow K_2(B, \mathfrak{M}))$  is not zero. But we have seen in diagram 3 that this group injects inside  $\text{Ker}(K_2(A) \rightarrow K_2(B))$ . This proves the theorem.  $\square$

**Proof of Corollary 1.3.** The corollary follows from Theorem 1.2 since

Berger's conjecture has been verified in these cases. For example, (i) and (ii) are verified in [CGW], (iii) in [B1], (iv) in [B2], (v) in [S], (vi) in [BA], (vii) in [GU], and (viii) in [HW].  $\square$

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